1 An introduction to homotopy theory

This semester, we will continue to study the topological properties of manifolds, but we will also consider more general topological spaces. For much of what will follow, we will deal with arbitrary topological spaces, which may, for example, not be Hausdorff (recall the quotient space $R_0 = \mathbb{R} \sqcup \mathbb{R}/(a \sim b \text{ iff } a = b \neq 0)$, or locally Euclidean (for example, the Greek letter θ), or even locally contractible (for example, the Hawaiian earring, given by the union of circles at $(1/n, 0)$ of radius $1/n$ for all positive $n \in \mathbb{Z}$ in $\mathbb{R}^2$ with the induced topology).

While we relax the type of space under consideration, we suitably relax the notion of equivalence which we are interested in: we will often be concerned not with homeomorphism (topological equivalence), but rather *homotopy equivalence*, which we recall now.

Definition 1. Continuous maps $f_0, f_1 : X \longrightarrow Y$ are *homotopic*, i.e. $f_0 \simeq f_1$, when there is a continuous map $F : X \times I \longrightarrow Y$, called a homotopy, such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. We sometimes write $F : f_0 \Rightarrow f_1$ to denote the homotopy.

The homotopy relation $\simeq$ is an equivalence relation: if $F_{01} : f_0 \Rightarrow f_1$ and $F_{12} : f_1 \Rightarrow f_2$ for maps $f_i : X \longrightarrow Y$, then

$$F_{02}(t, x) = \begin{cases} F_{01}(2t, x) & : \ 0 \leq t \leq 1/2 \\ F_{12}(2t - 1, x) & : \ 1/2 \leq t \leq 1 \end{cases}$$

gives a homotopy $F_{02} : f_0 \Rightarrow f_2$. Check reflexivity and identity yourself!

The homotopy relation is also compatible with the natural category structure on continuous functions: If $F : f_0 \Rightarrow f_1$ for $f_i : X \longrightarrow Y$, and $G : g_0 \Rightarrow g_1$ for $g_i : Y \longrightarrow Z$, then the composition

$$X \times I \xrightarrow{(F, \pi_I)} Y \times I \xrightarrow{G} Z$$

defines a homotopy $g_0 \circ f_0 \Rightarrow g_1 \circ f_1$. As a result of this, we may consider a new category, where the objects are topological spaces and the morphisms are *homotopy classes* of continuous maps.

Definition 2. Topological spaces, and homotopy classes of maps between them, form a category, **HTop**, called the *homotopy category of spaces*.

Because the notion of morphism is different in **HTop**, this changes the meaning of isomorphism – we are no longer dealing with homeomorphism.

Definition 3. Topological spaces X, Y are said to be *homotopy equivalent*(or *homotopic* or have the same homotopy type $X \simeq Y$) when they are isomorphic in the homotopy category. This means that there are maps $f : X \longrightarrow Y$, $g : Y \longrightarrow X$ such that $f \circ g \simeq \mathrm{Id}_Y$ and $g \circ f \simeq \mathrm{Id}_X$.

Example 1.1. *(Homotopy equivalences)*

- *The one-point space $\{*\}$ is homotopic to $\mathbb{R}$, since $* \mapsto 0$ and $x \mapsto * \ \forall x \in \mathbb{R}$ define continuous maps f, g which are homotopy inverses of each other. Similarly $\{*\} \simeq B^n \simeq \mathbb{R}^n \ \forall n$. Any space $\simeq *$ we call* contractible.
- *The solid torus $B^2 \times S^1$ is homotopic to S^1.*
- *Any vector bundle $E \longrightarrow X$ is homotopic to X itself.*
- *The "pair of pants" surface with boundary $S^1 \sqcup S^1 \sqcup S^1$ is homotopic to the letter θ.*

When considering maps $f : X \longrightarrow Y$, we may choose to consider equivalence classes of maps which are homotopic *only away from a distinguished subset $A \subset X$*. These are called homotopies *relative to A*.

Definition 4. For $f_i : X \longrightarrow Y$ and $A \subset X$, we say $F : f_0 \Rightarrow f_1$ is a homotopy rel A when $F(x,t) = f_0(x)$ for all $x \in A$ and for all t.

This is useful in the case that a space X can be "continuously contracted" onto a subspace $A \subset X$: we formalize this as follows:

Definition 5. A retraction of X onto the subspace $A \subset X$ is a continuous map $r : X \longrightarrow A$ such that $r|_A = \mathrm{Id}_A$. In other words, a retraction is a self-map $r : X \longrightarrow X$ such that $r^2 = r$, where we take $A = \mathrm{im}(r)$.

We then say that $A \subset X$ is a (strong) deformation retract of X when $\mathrm{Id}_X : X \longrightarrow X$ is homotopic (rel A) to a retract $r : X \longrightarrow A$.

Proposition 1.2. *If $A \subset X$ is a deformation retract of X, then $A \simeq X$.*

Proof. Take $r : X \longrightarrow A$ and the inclusion $\iota : A \longrightarrow X$. Then $\iota \circ r = r \simeq \mathrm{Id}_X$ by assumption, and $r \circ \iota = \mathrm{Id}_A$. Hence we have $X \simeq A$. ☐

Deformation retracts are quite intuitive and easy to visualize - they also can be used to understand any homotopy equivalence:

Proposition 1.3. *(See Hatcher, Cor. 0.21) X, Y are homotopy equivalent iff there exists a space Z containing X, Y and deformation retracting onto each.*

1.1 Cell complexes

The construction of a mapping cylinder M_f of a continuous map $f : X \longrightarrow Y$ is an example of the coarse type of gluing and pasting constructions we are allowed to do once we go beyond manifolds. In this section we will introduce more such constructions, and introduce a class of spaces which is very convenient for algebraic topology.

A *cell complex*, otherwise known as a CW complex, is a topological space constructed from disks (called *cells*), step by step increasing in dimension. The basic procedure in the construction is called "attaching an n-cell". An n-cell is the interior e^n of a closed disk D^n of dimension n. How to attach it to a space X? Simply glue D^n to X with a continuous map $\varphi : S^{n-1} \longrightarrow X$, forming:

$$X \sqcup D^n / \{x \sim \varphi(x) \ : \ x \in \partial D^n\}.$$

The result is a topological space (with the quotient topology), but as a set, is the disjoint union $X \sqcup e^n$.

Building a cell complex X

- Start with a discrete set X^0, whose points we view as 0-cells.
- Inductively form the n-skeleton X^n from X^{n-1} by attaching a set of n-cells $\{e^n_\alpha\}$ to X^{n-1}.
- Either set $X = X^n$ for some $n < \infty$, or set $X = \bigcup_n X^n$, where in the infinite case we use the weak topology: $A \subset X$ is open if it is open in X^n $\forall n$.

While cell complexes are not locally Euclidean, they do have very good properties, for example they are Hausdorff and locally contractible. Any manifold is homotopy equivalent to a cell complex.

Example 1.4. *The 1-skeleton of a cell complex is a graph, and may have loops.*

Example 1.5. *The classical representation of the orientable genus g surface as a $4g$-gon with sides identified cyclically according to $\ldots aba^{-1}b^{-1}\ldots$ is manifestly a cell complex with a single 0-cell, $2g$ 1-cells and a single 2-cell. One sees immediately from this representation that to puncture such a surface at a single point would render it homotopy equivalent to a "wedge" of $2g$ circles, i.e. the disjoint union of $2g$ circles where $2g$ points, one from each circle, are identified.*

Example 1.6. *The n-sphere may be expressed as a cell complex with a single 0-cell and a single n-cell. So $S^n = e^0 \sqcup e^n$.*

Example 1.7. *The real projective space $\mathbb{R}P^n$ is the quotient of S^n by the antipodal involution. Hence it can be expressed as the upper hemisphere with boundary points antipodally identified. Hence it is a n-cell attached to $\mathbb{R}P^n$ via the antipodal identification map. since $S^0 = \{-1, 1\}$, we see $\mathbb{R}P^0 = e^0$ is a single 0-cell, and $\mathbb{R}P^n = e^0 \sqcup e^1 \sqcup \cdots \sqcup e^n$.*

Note that in the case of $\mathbb{R}P^2$, the attaching map for the 2-cell sends opposite points of S^1 to the same point in $\mathbb{R}P^1 = S^1$. Hence the attaching map $S^1 \longrightarrow S^1$ is simply $\theta \mapsto 2\theta$. Compare this with the attaching map $\theta \mapsto \theta$, which produces the 2-disc instead of $\mathbb{R}P^2$.

Example 1.8. *The complex projective space, $\mathbb{C}P^n$, can be expressed as $\mathbb{C}^n$ adjoin the $n-1$-plane at infinity, where the attaching map $S^{2n-1} \longrightarrow \mathbb{C}P^{n-1}$ is precisely the defining projection of $\mathbb{C}P^{n-1}$, i.e. the generalized Hopf map. As a result, as a cell complex we have*

$$\mathbb{C}P^n = e^0 \sqcup e^2 \sqcup \cdots \sqcup e^{2n}.$$

1.2 The fundamental group(oid)

We are all familiar with the idea of connectedness of a space, and the stronger notion of path-connectedness: that any two points x, y may be joined by a continuous path. In this section we will try to understand the fact that there may be *different homotopy classes of paths connecting x, y*, or in other words[1], that the space of paths joining x, y may be disconnected.

To understand the behaviour of paths joining points in a topological space X, we first observe that these paths actually form a category: Define a category $\mathcal{P}(X)$, whose objects are the points in X, and for which the morphisms from $p \in X$ to $q \in X$ are the finite length paths joining them, i.e. define

$$\mathrm{Hom}(p, q) := \{\gamma : [0, \infty) \longrightarrow X \quad : \quad \exists R > 0 \text{ with } \gamma(0) = p, \quad \gamma(t) = q \quad \forall t \geq R\}.$$

We may then define the length of the path to be $T_\gamma = \mathrm{Inf}\{T \ : \ \gamma(t) = q \ \forall \ t \geq T\}$.

The composition

$$\mathrm{Hom}(p, q) \times \mathrm{Hom}(q, r) \longrightarrow \mathrm{Hom}(p, r)$$

is known as "concatenation of paths", which simply means that

$$(\gamma_2\gamma_1)(t) = \begin{cases} \gamma_1(t) & 0 \leq t \leq T_{\gamma_1} \\ \gamma_2(t - T_{\gamma_1}) & t \geq T_{\gamma_1} \end{cases}$$

The path category $\mathcal{P}(X)$ has a space of objects, X, and a space of arrows (morphisms), which is a subspace of $C^0([0, \infty), X)$. As a result, it is equipped with a natural topology: Take the given topology on X, and take the topology on arrows induced by the "compact-open" topology on $C^0([0, \infty), X)$. You can verify that the category structure is compatible with this topology.

[Recall: open sets in the compact-open topology are arbitrary unions of finite intersections of sets of the form $C^0((X, K), (Y, U))$, for $K \subset X$ compact and $U \subset Y$ open.]

Just as we simplified the category **Top** to form **HTop**, we can simplify our path category $\mathcal{P}(X)$ by keeping the objects, but considering two paths $\gamma, \gamma' \in \mathrm{Hom}(p, q)$ to be equivalent when they are homotopic rel boundary in the following sense[2]

Definition 6. Paths γ_0, γ_1 are *homotopic paths* (and we write $\gamma_0 \simeq \gamma_1$) when there is a homotopy

$$H : I \times [0, \infty) \longrightarrow X$$

such that $H(0, t) = \gamma_0(t)$ and $H(1, t) = \gamma_1(t)$ for all t, and there exists an $R > 0$ for which $H(s, 0) = p$ and $H(s, t) = q \ \forall t > R$, for all s.

[1] Consider this a heuristic statement - it is a delicate matter to compare $C^0(I \times I, X)$ and $C^0(I, C^0(I, X))$.

[2] We showed that homotopy is an equivalence relation; for the same reason, homotopy rel $A \subset X$ is, too.

Finally, note that if $\gamma_{pq} \in \mathrm{Hom}(p,q)$ and $\gamma_{qr} \in \mathrm{Hom}(q,r)$, and if we homotopically deform these paths $\gamma_{pq} \stackrel{h}{\Rightarrow} \gamma'_{pq}$ and$\gamma_{qr} \stackrel{k}{\Rightarrow} \gamma'_{qr}$, then the concatenation (and rescaling) of h and k gives a homotopy from $\gamma_{qr}\gamma_{pq}$ to $\gamma'_{qr}\gamma'_{pq}$. This shows that the category structure descends to homotopy classes of paths.

Modding out paths by homotopies, we obtain a new category, which we could call $H\mathcal{P}(X)$, but it is actually called $\Pi_1(X)$, the fundamental groupoid of X. Note that since $\mathcal{P}(X)$ has a topology from compact-open, then so does its quotient $\Pi_1(X)$. The reason it is called a groupoid is that it is a special kind of category: every morphism is *invertible*: given any homotopy class of path $[\gamma] : p \longrightarrow q$, we can form $[\gamma]^{-1} = [\gamma^{-1}]$, where $\gamma^{-1}(t) = \gamma(T_\gamma - t)$. Draw a diagram illustrating a homotopy from $\gamma^{-1}\gamma$ to the constant path p, proving that any path class is invertible in the fundamental groupoid.

Definition 7. The fundamental groupoid $\Pi_1(X)$ is the category whose objects are points in X, and whose morphisms are homotopy classes of paths[3] between points. It is equipped with the quotient of the compact-open topology.

From any groupoid, we can form a bunch of groups: pick any object $x_0 \in X$ in the category, and consider the space of all self-morphisms $\mathrm{Hom}(x_0, x_0)$ in the category. Since all morphisms in a groupoid are invertible, it follows that $\mathrm{Hom}(x_0, x_0)$ is a group – it is called the *isotropy group* of x_0. Since the fundamental groupoid has a natural topology for which the category structure is continuous, it follows that $\mathrm{Hom}(x_0, x_0)$ is a *topological group.*

Definition 8. The fundamental group of the pointed[4]space (X, x_0) is the topological group $\pi_1(X, x_0) := \mathrm{Hom}(x_0, x_0)$ of homotopy classes of paths beginning and ending at x_0.

In any groupoid, the isotropy groups of objects x, y are always isomorphic if $\mathrm{Hom}(x, y)$ contains at least one element γ, since the map $g \mapsto \gamma g \gamma^{-1}$ defines an isomorphism $\mathrm{Hom}(x, x) \longrightarrow \mathrm{Hom}(y, y)$ Therefore we obtain:

Proposition 1.9. *If $x, y \in X$ are connected by a path σ, then $\gamma \mapsto [\sigma]\gamma[\sigma]^{-1}$ defines an isomorphism $\pi_1(X, x) \longrightarrow \pi_1(X, y)$.*

Example 1.10. *Let X be a convex set in $\mathbb{R}^n$ (this means that the linear segment joining $p, q \in X$ is contained in X) and pick $p, q \in X$. Given any paths $\gamma_0, \gamma_1 \in \mathcal{P}(p, q)$, the linear interpolation $s\gamma_0 + (1 - s)\gamma_1$ defines a homotopy of paths $\gamma_0 \Rightarrow \gamma_1$. Hence there is a single homotopy class of paths joining p, q, and so $\Pi_1(X)$ maps homeomorphically via the source and target maps (s, t) to $X \times X$, and the groupoid law is $(x, y) \circ (y, z) = (x, z)$. This is called the* pair groupoid *over X. The fundamental group $\pi_1(X, x_0)$ is simply $s^{-1}(x_0) \cap t^{-1}(x_0) = \{(x_0, x_0)\}$, the trivial group.*

The final remark to make concerning the category $\mathcal{P}(X)$ of paths on a space X, and its homotopy descendant $\Pi_1(X)$, the fundamental groupoid, is that they depend functorially on the space X.

Proposition 1.11. *$\mathcal{P} : X \mapsto \mathcal{P}(X)$ is a functor from* **Top** *to the category of categories, taking morphisms (continuous maps) $f : X \longrightarrow Y$ to morphisms (functors) $f \circ - : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$. Furthermore, a homotopy $H : f \Rightarrow g$ defines a natural transformation $\mathcal{P}(f) \Rightarrow \mathcal{P}(g)$, and hence a homotopy equivalence $X \simeq Y$ gives rise to an equivalence of categories $\mathcal{P}(X) \simeq \mathcal{P}(Y)$.*

These properties descend to the fundamental groupoid, as well as to the fundamental group, implying that for any continuous map of pointed spaces $f : (X, x_0) \longrightarrow (Y, y_0)$, we obtain a homomorphism of groups $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$, given simply by composition $[\gamma] \mapsto [f \circ \gamma]$. This last fact is usually proven directly, since it is so simple.

[3]Note also that any path $\tilde{\gamma}$ may be reparamatrized via $\gamma(t) := \tilde{\gamma}(T_\gamma t)$ to a path of unit length, and that $\gamma' \simeq \gamma$, so that up to homotopy, only unit length paths need be considered (this is the usual convention when defining the fundamental group(oid)).

[4]A pointed space is just a pair $(X, A \subset X)$ where A happens to consist of a single point. Recall that pairs form a category, with $\mathrm{Hom}((X, A), (Y, B)) = \{f \in C^0(X, Y) \ : \ f(A) \subset B\}$.